On the hydrodynamical description of gauge theories

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Abstract. *Classes of gauge eguivalent configurations are characterized in terms of local gauge invariant quantities. The construction of such quantities is performed for scalar electrodynamics and theories of SU(2)-gauge fields in two different representations. Topologically nontrivial configurations are discussed.*

0. INTRODUCTION

It is well known that the configuration space Q of a gauge theory is built up by the set of connections on a principal fibre bundle. In this set acts the group $\mathcal{G}$ of local gauge transformations. Factorizing with respect to this action we obatin the reduced configuration space $Q/\mathcal{G}$ consisting of classes of gauge equivalent connections. A lot of hard and intersting mathematical probelms arise if one investigates the global structure of - speaking most generally - the stratification of the orbit space $Q/\mathcal{G}$, see e.g. [10], [11] and [12].

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Our aim is a more practical one. For purposes of quantum theory one has to parametrize $Q/\mathcal{G}$ in an effective way - one has to «fix the gauge». Unfortunately, usual methods of fixing the gauge (e.g. Lorentz-or Coloumb-gauge) destroy such properties of the theory like locality or relativistic invariance. (For a more detailed discussion of this point see [3]). In contrast to these methods we are going to parametrize classes of gauge equivalent configurations by the help of local gauge invariant quantities. In the case of scalar electrodynamics (chapter 1) our construction leads to the so called hydrodynamical quantities obtained earlier in [1] and [2]. In chapter 2 we demonstrate our method for two models with gauge group $SU(2)$ in different representations.

We should underline that our method works only for theories of gauge fields interacting with matter fields, but not for «pure gauge theories». Another point is that we are only able to perform our construction for generic (the exact meaning of this will be clear later on) configurations. Thus, we don't get any results concerning the global structure of $Q/\mathcal{G}$.

In some respects our treatment will be very sketchy. A more detailed discussion (including the reduction of the canonical structure of gauge theories in terms of hydrodynamical quantities) can be found in [3]. Throughout this paper we use the fibre bundle formulation of gauge theories as introduced in [8] and [9].

1. SCALAR ELECTRODYNAMICS

This theory describes the interaction of a $U(1)$-gauge field A with a complex scalar field φ. The Lagrangean is given by

$$L = -\mathcal{V}(\varphi\bar{\varphi}) + \frac{1}{2} D_\mu \varphi \overline{D^\mu \varphi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1.1 a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{1.1 b}$$

$$D_\mu \varphi = (\partial_\mu + \mathrm{ig}\, A_\mu) \quad . \tag{1.1 b}$$

In the language of fibre bundles a configuration of this theory is a pair (τ, f), where τ is a connection form in a (trivial) $U(1)$-principal bundle P over Minkowski space M and f is a section of the associated bundle $E = P \times_{U(1)} \mathbb{C}^1$, ($U(1)$ acts on $\mathbb{C}^1$ by left multiplication).

In a chosen trivialization

$$(\pi, \varkappa) : P \longrightarrow M \times U(1),$$

π-canonical projection in P, of P the pair (τ, f) is represented by a pair (A, φ) of quantities introduced above.

We shall characterize classes $[(\tau, f)]$ of gauge equivalent configurations by gauge invariant quantities. For this purpose let us consider a configuration (τ, f) and remove the submanifold $\sigma := \{x \in M : f(x) = 0\}$ from M. In the generic case σ is a 2-dimensional submanifold. We denote $M_0 := M - \sigma$ and observe that $\pi_1(M_0) \cong \mathbb{Z}$. Next we restrict P and E to subbundles P_0 and E_0 over M_0, and take the restrictions (denoted by the same letters) of τ resp. f to P_0 resp. E_0. Obviously

$$(1.2) \qquad M_0 \ni x \longrightarrow R(x) := \|f(x)\| \in \mathbb{R}^1_+$$

is a gauge invariant quantity and

$$(1.3) \qquad s(x) := f(x)\,(\|f(x)\|)^{-1}$$

defines a section of the subbundle $\widetilde{E}_0 = P_0 \times_{U(1)} S^1$ of E_0. It is trivial that $\widetilde{E}_0$ can be canonically identified with P_0 and, therefore, s defines a section (denoted by the same letter) of P_0. We use s to define a gauge invariant covector field v on M_0:

$$(1.4) \qquad \begin{aligned} v_{\pi(p)}(X) &:= \frac{1}{\mathrm{ig}}\, \psi_p^{-1}(\mathrm{ver}\, s_* X) \\ &= \frac{1}{i}\, \tau_p(s_* X), \quad \text{where } X \in T_{\pi(p)} M_0, \end{aligned}$$

$$\psi_p : u(1) \cong i\,\mathbb{R}^1 \longrightarrow V_p$$

is the canonical isomorphism of the Lie algebra of $U(1)$ and the vertical subspace V_p at $p \in P_0$, defined by the right action ψ of $U(1)$ on P_0, and $\mathrm{ver}\, s_* X$ is the vertical (with respect to τ) component of $s_* X$. Thus,

$$(1.5) \qquad v = \frac{1}{i} s^* \tau.$$

It is obvious that v is gauge invariant. The geometrical sense of this quantity is illustrated in Fig. 1:

Given a pair (v, R), we can reconstruct (τ, f) up to gauge transformations. Thus, we have a 1-1-correspondence

$$(1.6) \qquad [(\tau, f)] \longleftrightarrow (v, R).$$

In a chosen trivialization $(\pi, \varkappa)$ of P_0 we have

$$\tau = \pi^*(iA) + \frac{1}{e} \varkappa^*(\theta),$$

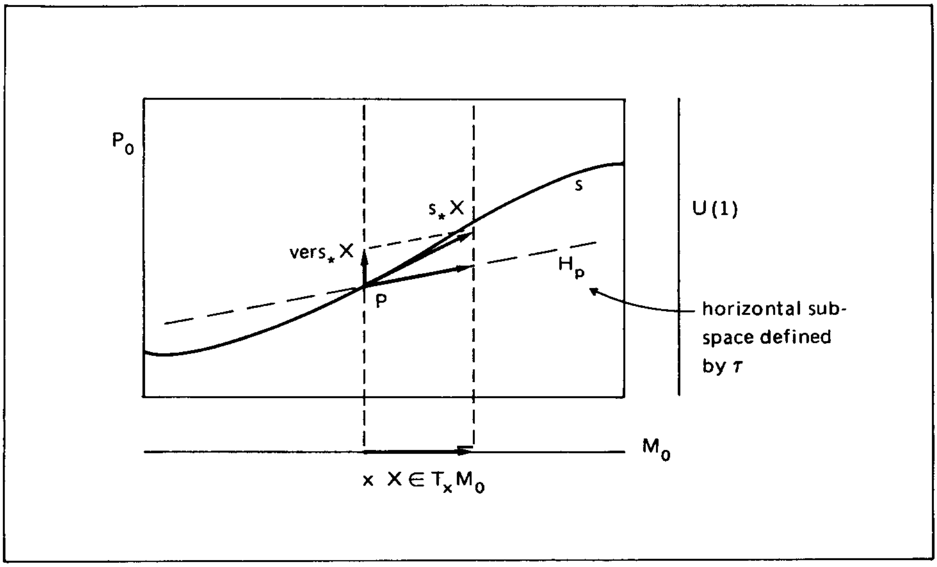

Fig. 1.

where θ is the canonical left invariant (Lie algebra-valued) 1-form on $U(1)$. Denoting $s(x) = \exp(i\alpha(x))$, we get

$$v = \frac{1}{i} s^{*}\tau = A + \frac{1}{e} \mathrm{d}\alpha. \tag{1.7}$$

(Of course, $\exp(i\alpha(x))$ is the phase of the matter field in the trivialization $\varkappa$).

The quantities (v, R) were earlier introduced by Biaıynicki-Birula ([1]) and Mandelstam ([2]). It appears that the field equations written down in terms of R and v are equivalent to the equations of relativistic hydrodynamics of a charged fluid, where R^2 is interpreted as the density and v as the velocity field of the fluid.

At the end let us notice that v has to fulfill certain topological constraints - due to the existence of topological nontrivial configurations (vortices) appearing in this model:

$$\operatorname{curl} v = F + 2\pi n e^{-1} \delta^{(2)}, \qquad \text{where } n \in \mathbb{Z} \tag{1.8}$$

is the vortex strength and $\delta^{(2)}$ is a δ-distribution concentrated on the vortex--submanifold σ. To show (1.8) we take a closed curve γ in M_0 and consider the phase-change of φ along γ. We obtain a closed curve in $U(1)$, which defines an element n of $\pi_1(U(1)) \cong \mathbb{Z}$. Obviously, the total phase-increase of φ along γ si $2\pi n$. Using this fact and (1.7), we get (1.8).

2. HYDRODYNAMICAL DESCRIPTION OF MODELS WITH GAUGE GROUP SU(2)

2.1. SU(2) in the fundamental representation

In this case we obtain a model describing the (minimal-coupling) interaction of a $SU(2)$-gauge field A with a $\mathbb{C}^2$-valued matter field φ. Thus, in our geometrical approach a configuration of this model is a pair (τ, f), where τ is a connection form in a (trivial) $SU(2)$-principal bundle P and f is a section of the associated bundle $E = P \times_{SU(2)} \mathbb{C}^2$. Now, $\sigma := \{x \in M : f(x) = 0\}$ is in the generic case a point, We take again $M_0 = M - \sigma$, $\pi_3(M_0) \cong \mathbb{Z}$, the corresponding restrictions of P and E to (trivial) bundles P_0 and E_0 over M_0, and also the restrictions of τ resp. f to P_0 resp. E_0. Of course,

$$M_0 \ni x \longrightarrow R(x) = \|f(x)\| \in R^1_+$$

is a gauge invariant quantity and

$$s(x) := f(x)\,(\|f(x)\|)^{-1} \qquad \text{is a section}$$

of the subbundle $\widetilde{E}_0 = P_0 \times_{SU(2)} S^3$ of E_0.

Again, $\widetilde{E}_0$ can be canonically identified with P_0 and, therefore, s defines a section of P_0. Our (gauge-invariant) «hydrodynamical velocity»

$$v := s^*\tau$$

is now a covector field on M_0 with values in the Lie algebra $su(2)$ and has essentially the same geometrical interpretation as in chapter 1. Similar topological constraints for v follow from the existence of nontrivial group homomorphisms

$$\pi_3(M_0) \cong \mathbb{Z} \longrightarrow \pi_3(S^3) \cong \mathbb{Z}.$$

2.2. SU(2) in the adjoint representation

Now we consider the theory of a $SU(2)$-gauge field A interacting with a 3-component matter field φ, defined by the Lagrangean:

$$L = -\mathcal{V}(\|\varphi\|^2) + \frac{1}{2}\|D\varphi\|^2 - \frac{1}{4}\|F\|^2, \tag{2.4 a}$$

where $\|\cdot\|$ is calculated by the help of the scalar product $h(\cdot\,,\cdot) = -\frac{1}{2}K(\cdot\,,\cdot)$, ($K$ - Cartan-Killing-form), on $su(2)$ and by the space time metric $g_{\mu\nu}$. It is easy to see that $e_a = -\frac{i}{2}\sigma_a$, ($\sigma_a$) - Pauli-matrices, is an orthonormal basis for h and that

(2.4)
$$\text{b)}\quad D_\mu \varphi^a = \partial_\mu \varphi^a + g\epsilon^a_{bc} A^b_\mu \varphi^c,$$
$$\text{c)}\quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^a_{bc} A^b_\mu A^c_\nu.$$

Moreover, under the identification

$$su(2) \cong \mathbb{R}^3,$$

the adjoint representation Ad of $SU(2)$ is isomorphic to the fundamental representation O of $SO(3)$ on $\mathbb{R}^3$:

$$Ad\, \rho(x^a e_a) = (O\, x)^a e_a, \qquad x \in R^3. \tag{2.5}$$

We take again a configuration (τ, f), where τ is a connection form in the (trivial) $SU(2)$-principal bundle P over M and f is a section of the associated vector bundle $E = P \times_{SU(2)} su(2)$. The submanifold σ defines in the generic case a line, thus for $M_0 = M - \sigma$ we have $\pi_2(M_0) = \mathbb{Z}$. After the same procedure as previously we end up with a pair (τ, f) in bundles P_0 and E_0 over M_0. Again

$$M_0 \ni x \longrightarrow R(x) := \| f(x) \| R^1_+ \tag{2.6}$$

is gauge invariant and now

$$s(x) := f(x)\, (\| f(x) \|)^{-1} \tag{2.7}$$

defines a section of the subbundle $\widetilde{E}_0 = P_0 \times_{SU(2)} S^2$ of E_0. But, of course, P_0 cannot be identified with $\widetilde{E}_0$ and, therefore, s does not define a section of P_0, but a subbundle $\hat{P}_0$.

Fixing $e_3 \in S^2$ and taking its stabilizer Stab $(e_3) \cong SO(2)$, we obtain - using (2.5) - an embedding

$$U(1) \hookrightarrow SU(2). \tag{2.8}$$

Factorizing with respect to the - induced by (2.8) - $U(1)$-action we get a principal $U(1)$-bundle

$$\chi : P_0 \longrightarrow P_0/U(1).$$

It is easy to see that $P_0/U(1) = \widetilde{E}_0$. Now, the section (2.7) defines an (in general nontrivial) $U(1)$-subbundle $\hat{P}_0 \subset P_0$ in the following way ([4]):

$$\hat{P}_0 := \{ p \in P_0 : \chi(p) = s(\pi(p)) \}. \tag{2.9}$$

Taking the (orthogonal with respect to h) decomposition

$$su(2) = u(1) \oplus u(1)^\perp, \tag{2.10}$$

induced by (2.8), we have

PROPOSITION: *a) The restriction*

(2.11 a) $\hat{\tau} := \tau^{u(1)}|_{P_0}$

of the $u(1)$-component of τ to $\hat{P}_0$ is a connection form on $\hat{P}_0$.

b) The restriction

(2.11 b) $\hat{\vartheta} := \tau^{u(1)^{\perp}}|_{P_0}$

of the $u(1)^{\perp}$-component of τ to $\hat{P}_0$ is a horizontal (with respect to $\hat{\tau}$) 1-form on $\hat{P}_0$ with values in $u(1)^{\perp} \cong \mathbb{C}^1$ of type $Ad|_{U(1)}$.

Proof. a) see [4], p. I, prop. 6.4.

b) by a simple calculation

Thus, in a first step we have characterized (τ, f) by a triple $(\hat{\tau}, \hat{\vartheta}, R)$, where $\hat{\tau}$ is a $U(1)$-gauge field and $\hat{\vartheta}$ a $\mathbb{C}^1$-valued covector (matter) field.

As we already mentioned, $\hat{P}_0$ will be in general a nontrivial bundle. This is due to the existence of topologically nontrivial configurations (monopoles), giving rise to nontrivial group homomorphisms $\pi_2(M_0) \cong \mathbb{Z} \longrightarrow \pi_2(S^2) \cong \mathbb{Z}$. An example (monopole of strength n on the time axis) is given by

$$M \ni x \longrightarrow \varphi(x) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}(x), \quad \text{where}$$

$$\varphi_1(x) + i\varphi_2(x) = (a_1 x_1 + i a_2 x_2)^n \quad \text{and} \quad \varphi_3(x) = a_3 x_3 .$$

For a detailed discussion of magnetic monopoles see [5], [6] and [7].

Now we choose a local trivialization of P_0, such that the section (2.7) is represented by

$$M_0 \ni x \longrightarrow s(x) \equiv e_3 \in S^2$$

and denote the representative of τ in this trivialization by A. It is easy to show that $B \equiv A^3$ resp. $V \equiv A^1 + iA^2$ are then representatives of $\hat{\tau}$ resp. $\hat{\vartheta}$. Using this, we may write down the Lagrangean (2.4a) after this first step of reduction:

$$L = R_\mu R^\mu + g^2 R^2 V_\mu \bar{V}^\mu - \mathcal{V}(R^2) - \frac{1}{4}(W_{\mu\nu}\bar{W}^{\mu\nu} + G_{\mu\nu}G^{\mu\nu}), \tag{2.12}$$

$$W_{\mu\nu} = D_{[\mu} V_{\nu]} = \partial_{[\mu} V_{\nu]} + ig\, B_{[\mu} V_{\nu]},$$

$$G_{\mu\nu} = \partial_{[\mu} B_{\nu]} - g \,\mathrm{Im}\,\{V_\mu \bar{V}_\nu\} \quad \text{and} \quad R_\mu = \partial_\mu R.$$

In a last step we have to parametrize the class $[(\hat{\tau}, \hat{\vartheta})]_{U(1)}$. For this purpose we decompose $\hat{\vartheta} = \hat{\vartheta}_1 + i\,\hat{\vartheta}_2$ and observe that the covectors $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ span a 2--dimensional plane $\mathscr{P}$ in every horizontal (with respect to $\hat{\tau}$) subspace of $T_{\hat{p}}\hat{P}_0$. Gauge transformations rotate $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ in $\mathscr{P}$. More exactly.

The endpoints of $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ draw in $\mathscr{P}$

1) an ellipse - if $\mathscr{P}$ is (with respect to the in $T_{\hat{p}}\hat{P}_0$ from M_0 induced scalar product) space-like,

2) a hyperbole - if $\mathscr{P}$ is time-like,

3) a straight line - if $\mathscr{P}$ is light-like.

Now, there exists a gauge, which from the geometrical point of view seems to be most natural: the gauge, in which the two covectors are orthogonal (with respect to the induced scalar product). Performing this gauge transformation, we get

$$\text{a)}\quad \hat{\tau} \longrightarrow \hat{\tau}',$$

$$\text{b)}\quad \hat{\vartheta} \longrightarrow \hat{\vartheta}', \quad (\hat{\vartheta}_1', \hat{\vartheta}_2') = 0.$$

(For case 3) this construction doesn't make sense!).

Thus, the class $[(\hat{\tau}, \hat{\vartheta})]_{U(1)}$ will be parametrized by $\hat{\tau}'$ and seven independent fields describing the geometry of the ellipse (or hyperbole).

One can show that transformation (2.13) is not well defined if simultaneously $(\hat{\vartheta}_1, \hat{\vartheta}_2) = 0$ and $\| \vartheta_1 \| = \| \vartheta_2 \|$. In the generic case these two equations define a 2-dimensional submanifold of M_0, which one has to remove from M_0 in order to perform the above construction.

In our opinion it would be interesting to perform similar constructions for theories containing spinor fields. For the case of spinor electrodynamics this has been already done, see [13], [14] and [15].

REFERENCES

[1] I. BIALYNICKI-BIRULA, *Relativistic Quantum Mechanics of Dyons. Exact Solution*, Phys. Rev. D, Vol. 3, No. 10, (1971), 2413 - 2415.

[2] S. MANDELSTAM, *Quantum Electrodynamics without Potentials*, Ann. of Phys., 19, (1962), 1 - 25.

[3] J. KIJOWSKI, G. RUDOLPH, *Hydrodynamical description of theories of gauge fields interacting with matter fields*, Rep. on Math. Phys., Vol. 20, (1981), 139 - 159.

[4] S. KOBAYASHI, K. NOMIZU, Foundations of differential geometry, Vol. 1, New York 1963.

[5] G. 't HOOFT, *Magnetic monopoles in unified gauge theories*, Nucl. Phys. B 79, (1974), 276 - 284.

[6] J. ARAFUNE, P.G.O. FREUND, C.J. GOEBEL, *Topology of Higgs fields*, Journ. of Math. Phys., Vol. 16, 2, (1975), 433 - 437.

[7] L.J. BOYA, J.F. CARINENA, J. MATEOS, *Homotopy and Solitons*, Fortschr. d. Phys., Bd. 26, No. 3, (1978), 175 - 214.

[8] A. TRAUTMAN, *Fibre bundles associated with space time*, Rep. on Math. Phys., Vol. 1, (1970), 29 - 50.

[9] A. TRAUTMAN, *Elementary introduction to fibre bundles and gauge fields*, prepr. of the Warsaw Univ. IFT/10/78, (1978).

[10] P.K. MITTER, C.M. VIALLET, *On the bundle of Connections and the gauge orbit Manifold in Yang-Mills Theory*, Comm. Math. Phys., Vol. 79, No. 4, (1981), 457 - 472.

[11] I.M. SINGER, *Some remarks on the Gribov Ambiguity*, Comm. Math. Phys., Vol. 60, No. 1, (1978), 7 - 12.

[12] W. KONDRACKI, J. ROGULSKI, *On the stratification of the orbit space for the action of automorphisms on connections*, prepr. 62/83/160, Inst. of Math., PAN, Warsaw, (1983), 1 - 110.

[13] T. TAKABAYASI, *Relativistic Hydrodynamics of the Dirac Matter*, Suppl. of the Progr. of Theor. Phys., No. 4, (1957), 1 - 80.

[14] J. MICKELSSON, *On the unification of electroweak interactions with gravity*, Czech. Journ. of Physics, Vol. B 32, (1982), 521 - 528.

[15] A. JAKUBIEC, *On a hydrodynamical description of the Dirac equation*, to appear in Lett. in Math. Phys.

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